8. Advanced Methods

Model Reduction of Linear Time Invariant Systems

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Outline

Error Estimate for Moment Matching

Parametric Model Reduction

Iterative Grammian-based methods

- **Global Error Estimate** $$\|G - G_{red}\|_\infty < 2(\sigma_{r+1} + \ldots \sigma_N)$$

- **Local Error Estimate** $$|G(s) - G_{red}(s)| \approx |G_{red,i}(s) - G_{red,i-1}(s)|$$

- **Difference between *i* and *i*-1 reduced model is about the difference between *i*-th and the full model.**

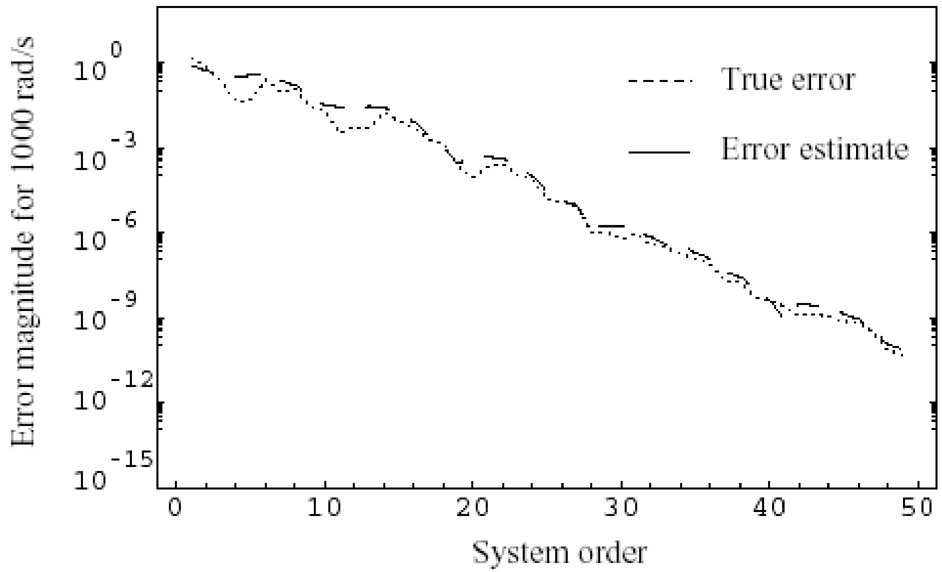
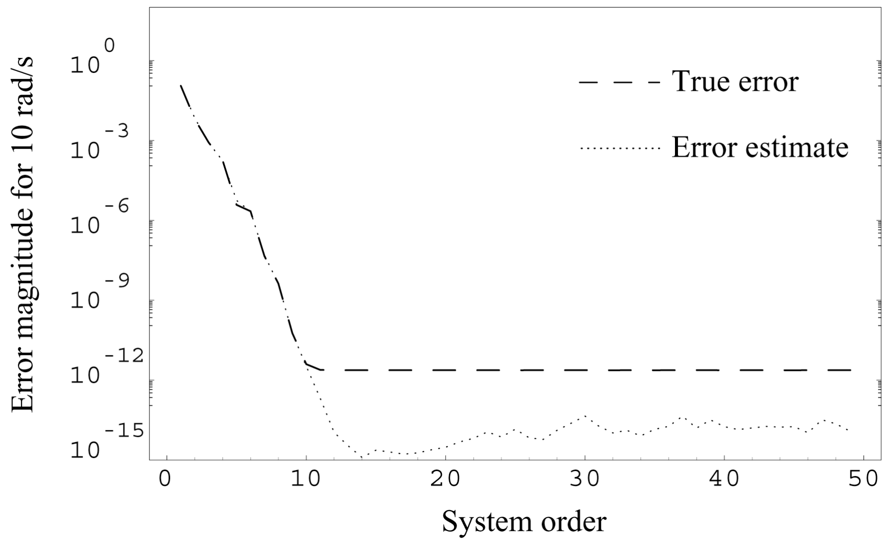
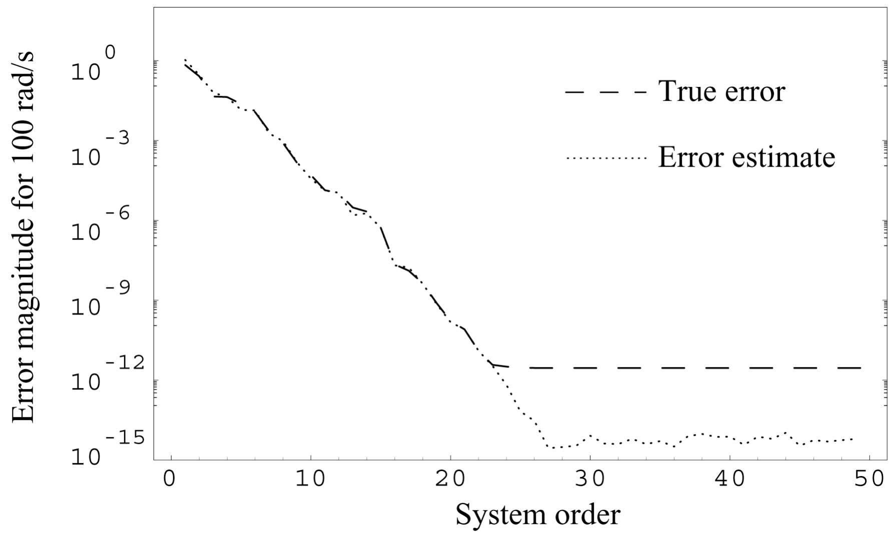
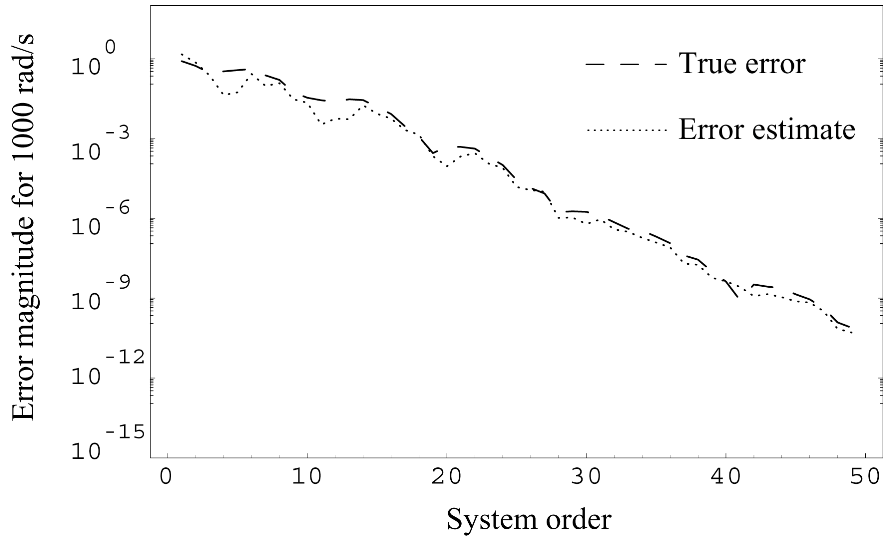


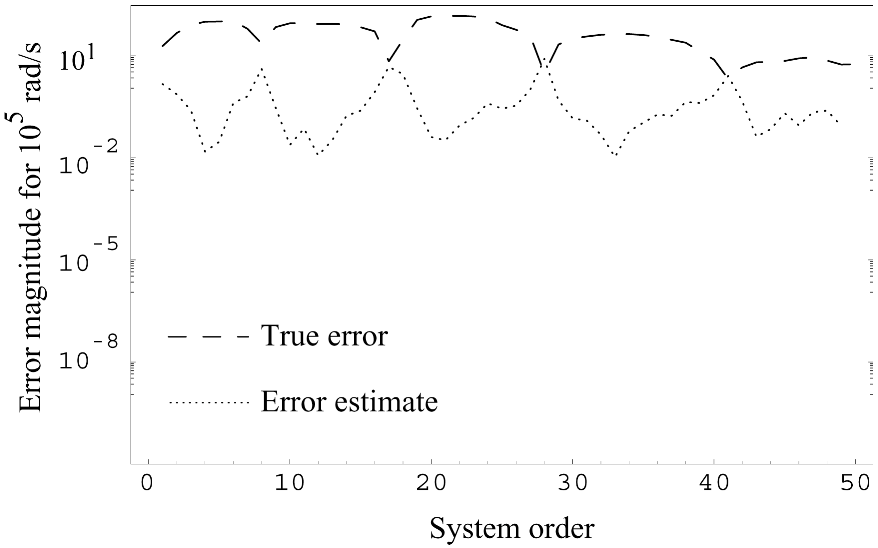
Fig. 5 Error estimate for $\omega = 10^3\ rad/s$.

Frequency-Domain Estimate for Microthruster



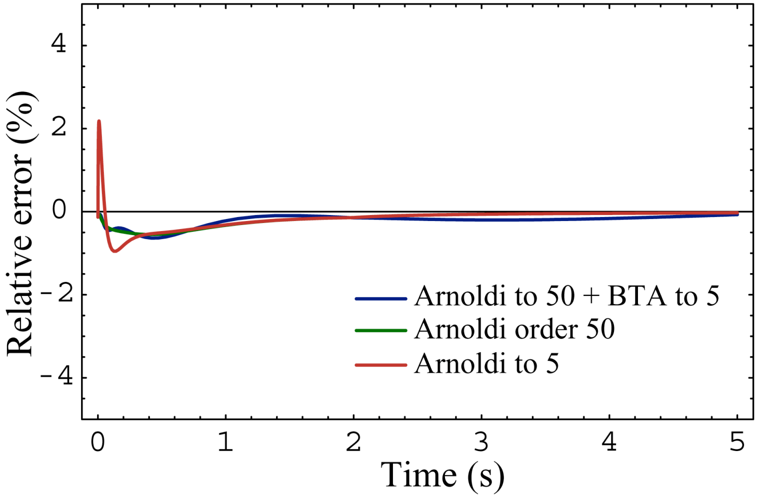




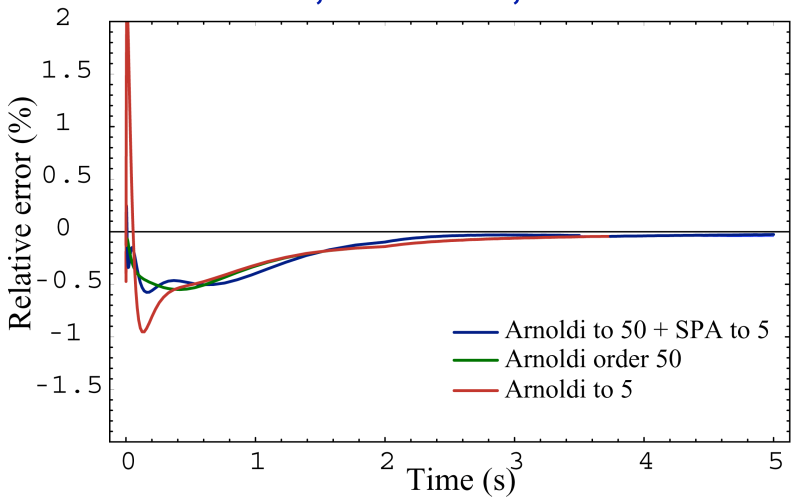


Sequential: Arnoldi followed by Control theory

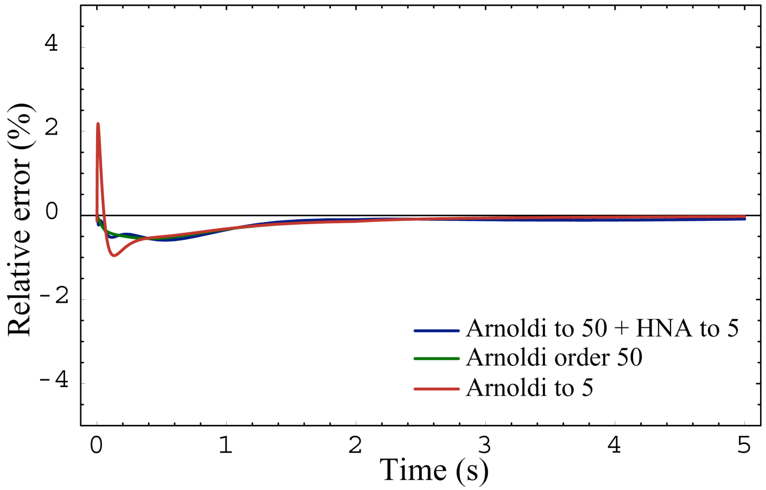
BTA, r1 = 50, r2 = 5



SPA, r1 = 50, r2 = 5



HNA, r1 = 50, r2 = 5



Parametric Model Reduction

- It would be good to preserve some properties as parameters.
- Natural for mixed boundary conditions and material properties.
- Can include geometry but it is much more work.
- Projection formalism is working for linear systems.

$$E\dot{x} = Ax + Bu$$
$$y = Cx$$

$$A = A_0 + \sum_i k_i A_i$$

$$V^T A V = V^T A_0 V + \sum_i k_i V^T A_i V$$

Multivariate Padé Approximation

- *s* does not have to be frequency.
- Can be a design parameter.
- Model reduction for stationary problem.
- Make multivariate Tailor expansion and match multivariate moments.

$$G(s) = C(sE - A)^{-1} B$$

$$P(s)x = b$$

$$G(s, k_i) = C\left(sE - A_0 + \sum_i k_i A_i \right)^{-1} B$$

Experience with Parametric MOR

- **Ms Feng has developed two methods how to find a good *V*.**
- **There is a paper from Nakhla's group.**
- **There is a paper from White's group.**
- **Many questions to research.**
- **Exciting opportunities:**
  - Boundary Condition Independent Model Reduction
  - Preserving material properties: Inverse problem
  - One can insert nonlinearity after model reduction

$$V^T A V = V^T A_0 V + \sum_i k_i V^T A_i V$$

Example from Ms Feng

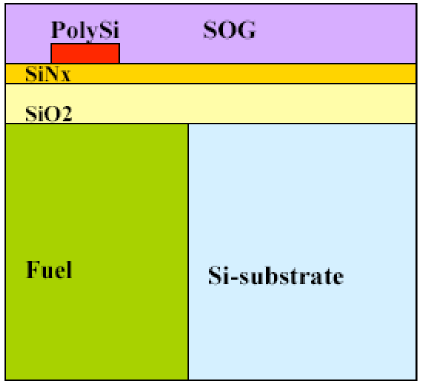
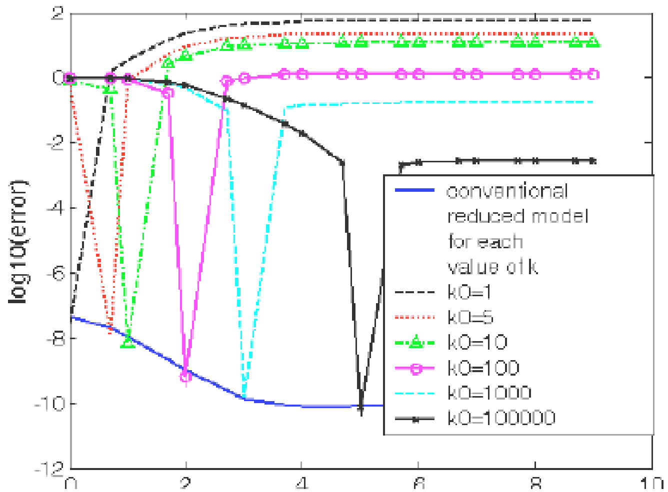


Fig. 1. A 2D-axisymmetrical model of the microthruster unit (not scaled). The axis of the symmetry on the left side. A heater is shown by a red spot.



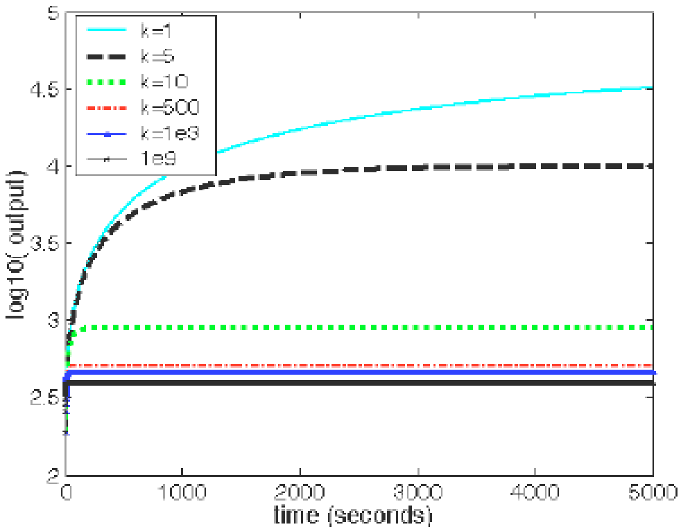
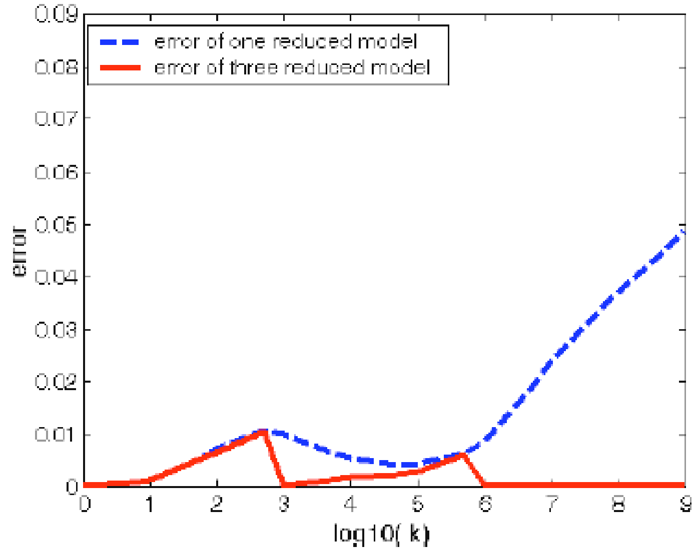


Fig.2.a Output at node 1 with different values of *k*



Decay of Hankel Singular Values

$$\|G - G_{red}\|_\infty < 2(\sigma_{r+1} + \ldots \sigma_N)$$

- How to estimate the small HSV?
- Worst-case scenario does not work.
- Penzl's estimate is based o maximum and minimum eigenvalues of the system matrix.
- Antoulas' estimate is based on the whole spectrum.

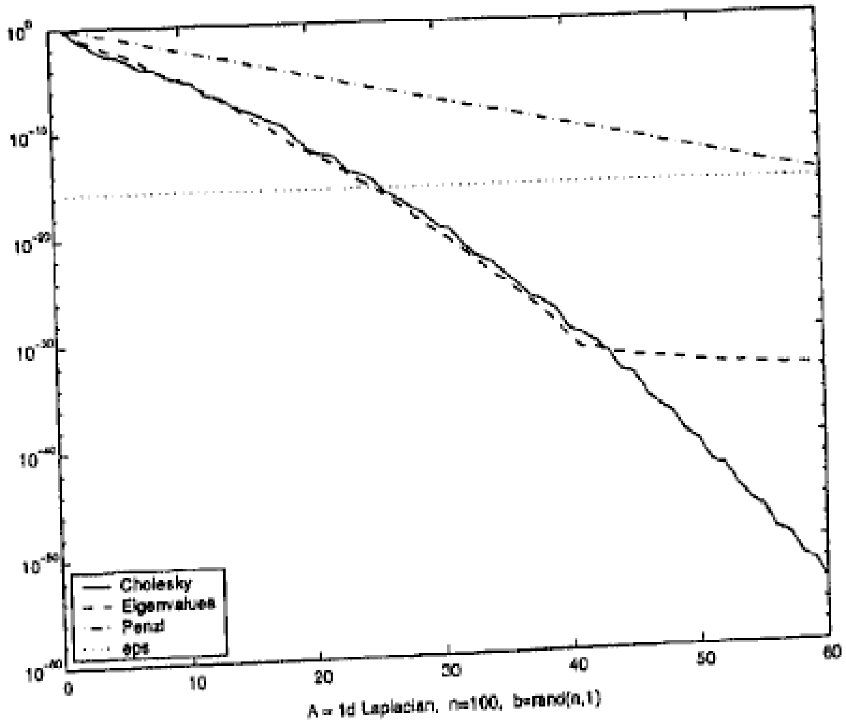


Fig. 2. Comparison of estimates: discrete Laplacian.

from Antoulas

Solving Lyapunov Equations by Iterative Methods

- Lyapunov equations can be expressed as a normal linear system of order N^2.
- One can apply iterative methods by making use of a special form for such a system.
- See chapter in Datta's book.

$$AX + XB = C$$

$$Gx = c$$

$$G = A \otimes I + I \otimes B^T$$

Low-rank Grammian approximation

- **Penzl; Lee and White; Gugercin, Sorensen and Antoulas.**
- **Express Grammian as** $P = XX^T$
- **Substitute into the Lyapunov equations and find an iterative method.**
- **Software LYAPACK, www.netlib.org/lyapack**
- **Problems:**
  - there are two Lyapunov equations to solve
  - model reduction theory for symmetric systems
  - may not preserve stability

$$AP + PA^T = -BB^T \qquad A^T Q + QA = -C^T C$$

Connection to moment matching

- Theorem from Jing-Rebecca Li for symmetric systems.
- Low-rank Grammian approximation is equivalent to multi-point expansion.
- Gives us some approximate theory how to choose expansion points.
- Input: maximum and minimum eigenvalues of the system matrix and tolerance.
- Computing elliptic integrals.
- Output: number and values of expansion points.

Sylvester equation

- Sorenson and Antoulas: model reduction based on the Sylvester equation.
- Valid for symmetric transfer function matrices.
- SISO is always appropriate.
- Can always be done for an arbitrary MIMO system.

$$AP + PA^T = -BB^T$$

$$A^T Q + QA = -C^T C$$

$$AR + RA = -BC$$

cross-grammian