6. Model Reduction via Projection

Model Reduction of Linear Time Invariant Systems

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Model Reduction via Truncation

Projection = Transformation + Truncation

Low Dimension Subspace

How to Choose Projection

Control Theory Methods

Truncation

- Choose important states
- Reorganize the matrices
- Discard unimportant states

$$\begin{vmatrix}\dot{z}_1\\ \dot{z}_2\end{vmatrix} = \begin{vmatrix}\bar{A}_{11} & \bar{A}_{12}\\ \bar{A}_{21} & \bar{A}_{22}\end{vmatrix}\begin{vmatrix}z_1\\ z_2\end{vmatrix} + \begin{vmatrix}\bar{B}_1\\ \bar{B}_2\end{vmatrix}u$$

$$y = [\bar{C}_1 \quad \bar{C}_2]\begin{vmatrix}z_1\\ z_2\end{vmatrix}$$

- Make sense only after appropriate transformation: Model Reduction = Transformation + Truncation

$$\dot{z}_1 = \bar{A}_{11}z_1 + \bar{B}_1 u$$

$$y = \bar{C}_1 z_1$$

- Overbar means transformed matrices

$$x = Uz$$

Transformation + Truncation = Projection

- It would be good not to compute the transformation explicitly.
- Let us combine this in a single step.
- The meaning of U_1^{-1} is pure operational.

$$x = [U_1 \quad U_2]\begin{vmatrix} z_1 \\ z_2 \end{vmatrix}$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = U^{-1}x = \begin{bmatrix} U_1^{-1} \\ U_2^{-1} \end{bmatrix} x$$

$$\bar{A} = U^{-1}AU$$

$$\downarrow$$

$$\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \begin{bmatrix} U_1^{-1} \\ U_2^{-1} \end{bmatrix} A [U_1 \quad U_2] \longrightarrow$$

$$\bar{A}_{11} = U_1^{-1} A U_1$$

$$\bar{B}_1 = U_1^{-1} B$$

$$\bar{C}_1 = C U_1$$

Low Dimension Subspace

- Since this slide *z* means low dimensional subspace only.

$$x = [U_1 \quad U_2] \begin{vmatrix} z_1 \\ z_2 \end{vmatrix}$$

- Low dimensional subspace should approximate *x* as a function of time or frequency.

$$x = U_1 z_1 + U_2 z_2 = Vz + \varepsilon$$

- Left subspace may be different from the right one.

$$\min \int \{x(t) \int Vz(t)\} dt$$

Orthogonal Projection

- **Orthonormal basis simplifies things.**

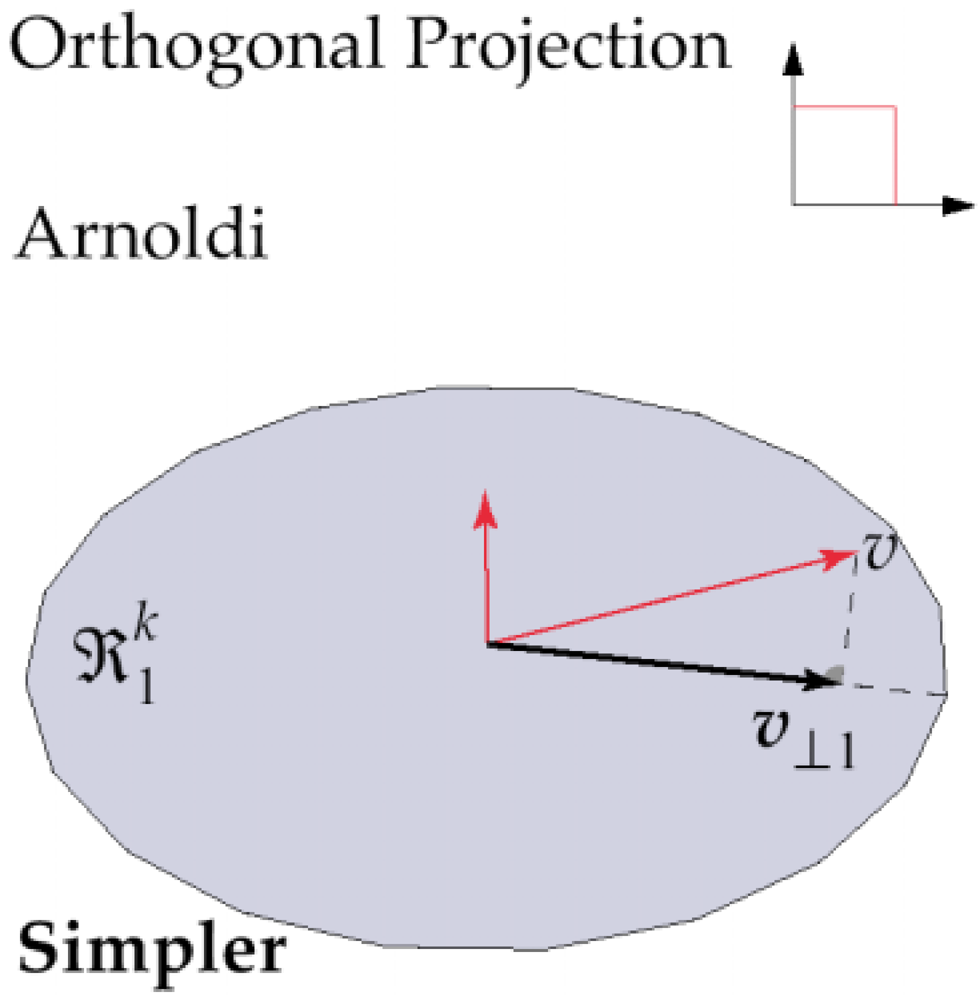
$$U^{-1} = U^T$$

$$U_1^{-1} = U_1^T$$

$$V^{-1} = V^T$$

- **Congruent transform preserves definiteness.**

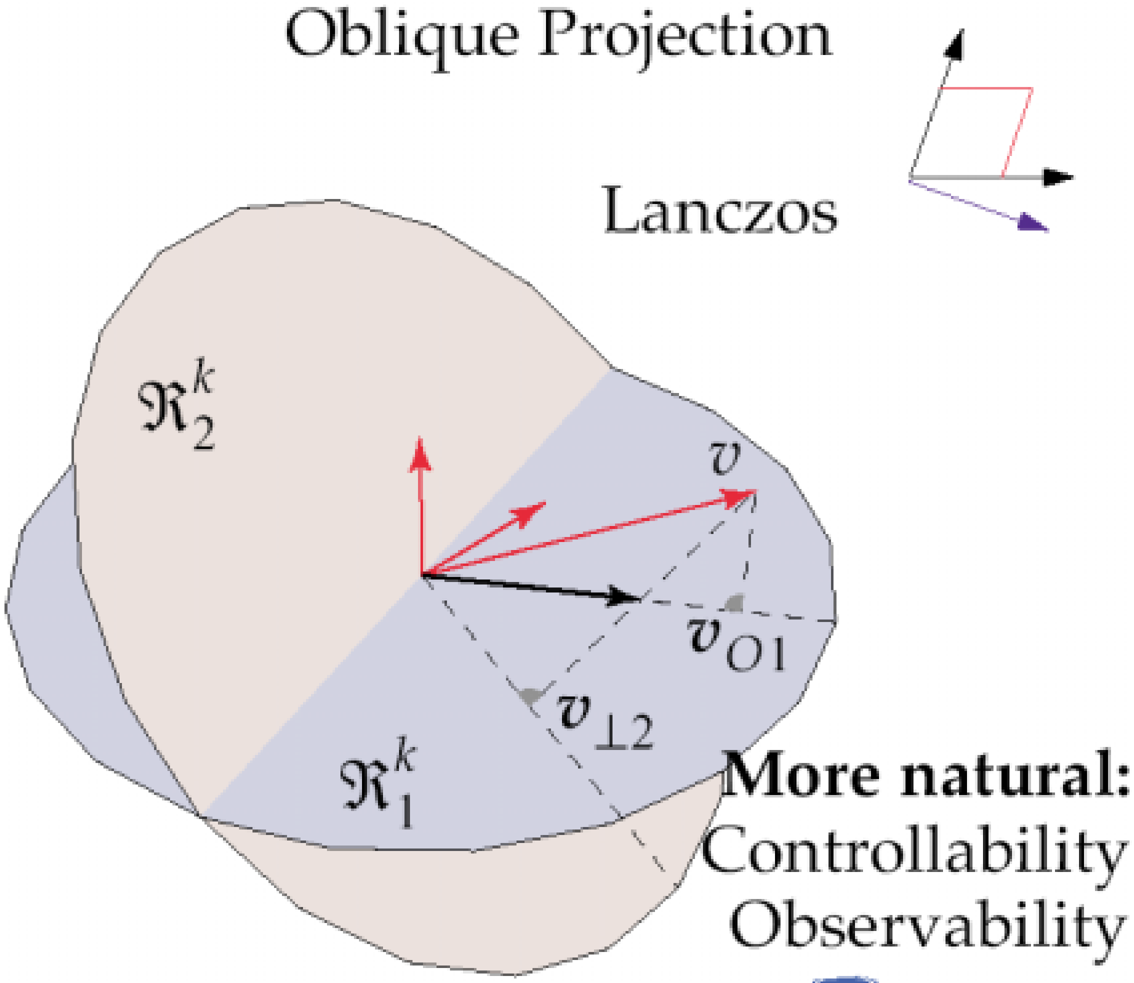
$$A_r = V^T A V$$



Oblique Projection

Bi-orthogonal basis:

$$W^T V = I$$



Projecting Other Representations

- Natural extension to non-canonical forms.

$$E\dot{x} = Ax + Bu$$
$$y = Cx$$

$$x = Vz$$

$$V^T E V \dot{z} = V^T A V z + V^T B u$$
$$y = CVz$$

- An alternative is the transformation to the canonical form and then projection.

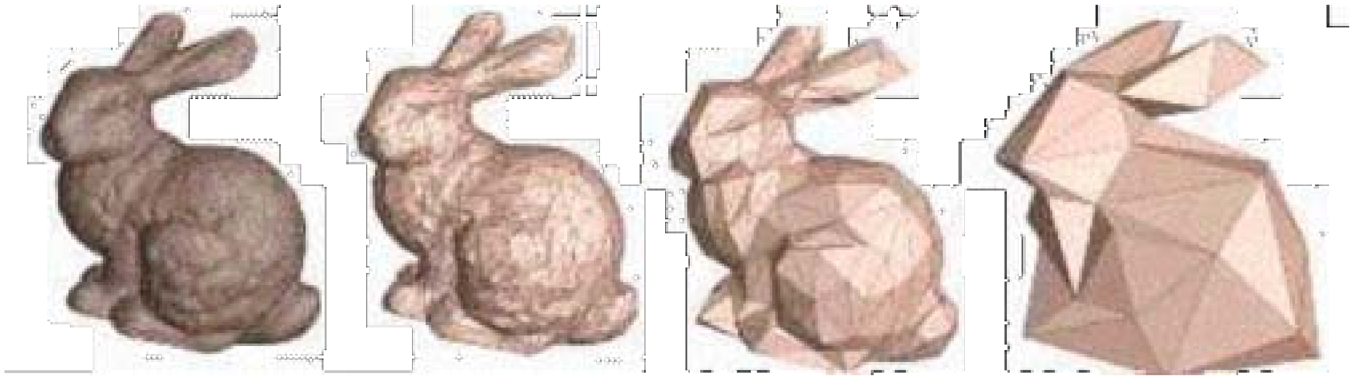
$$\dot{x} = E^{-1} A x + E^{-1} B u$$
$$y = Cx$$

- May produce different reduced models.

$$\dot{z} = V^T E^{-1} A V z + V^T E^{-1} B u$$
$$y = CVz$$

Coarsening in Physical Space

$$x = Vz$$



http://www.win.tue.nl/macsi-net/WG/WG2.html

- Adaptive meshing
- Preserves geometrical relationship
- Based on functions with local basis support
- Opportunity for model reduction is limited

Coarsening in Functional Space

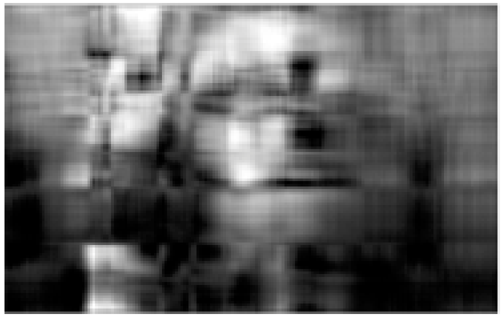
$$x = Vz$$

$$A = U\sigma V^T = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \ldots + \sigma_n u_n v_n^T$$

- Low-rank approximation.
- Columns of V are global functions.
- Values of *z* are dimensionless amplitudes.

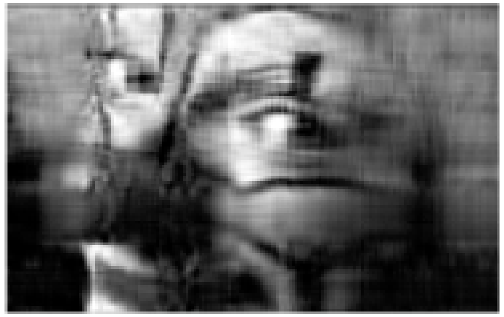
Clown: original picture

Clown: rank 6 approximation

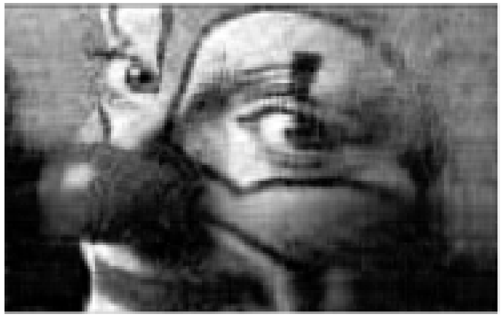


pictures from Antoulas

Clown: rank 12 approximation



Clown: rank 20 approximation



Overview of Methods

Table 3: Methods of model reduction of linear dynamics systems

Name	Advantages	Disadvantages
Control theory (Truncated Balanced Approximation, Singular Perturbation Approximation, Hankel-Norm Approximation)	Have a global error estimate, can be used in a fully automatic manner.	Computational complexity is $O(N^3)$, can be used for systems with order less than a few thousand unknowns.
Padé approximants (moment matching) via Krylov subspaces by means of either the Arnoldi or Lancsoz process.	Very advantageous computationally, can be applied to very high-dimensional 1st order linear systems.	Does not have a global error estimate. It is necessary to select the order of the reduced system manually.
SVD-Krylov (low-rank Grammian approximants).	Have a global error estimate and the computational complexity is less than $O(N^2)$.	Just under development.

Hankel Singular Values (HSV)

- System: $\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & \end{array}\right]$

- Impulse response:

$$h(t) = C \cdot e^{At} \cdot B$$

- Input-to-state: $\xi(t) = e^{At} \cdot B$

- State-to-output: $\eta(t) = C \cdot e^{At}$

- Grammians:

$$P = \int_0^\infty (e^{At} \cdot B \cdot B^T \cdot e^{A^T t}) dt$$

$$Q = \int_0^\infty (e^{A^T t} \cdot C^T \cdot C \cdot e^{At}) dt$$

- Lyapunov equations:

$$A \cdot P + P \cdot A^T + B \cdot B^T = 0$$

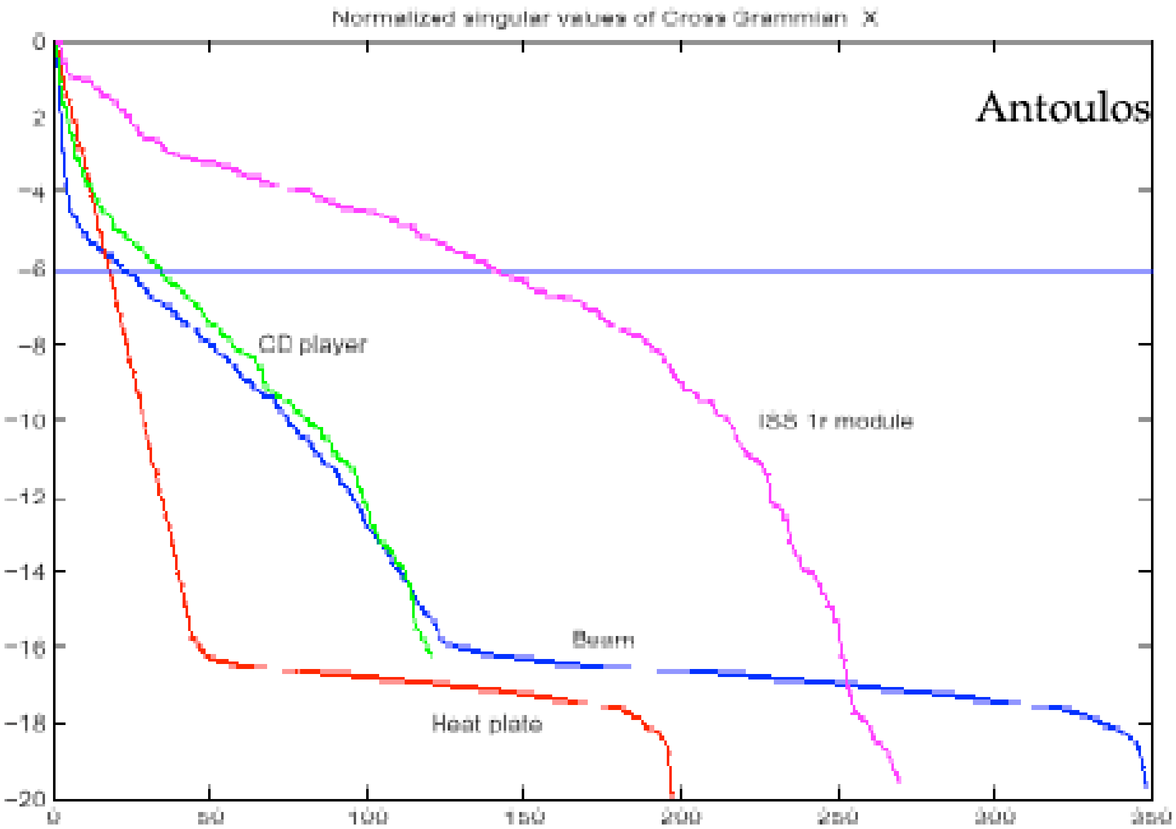
$$A^T \cdot Q + Q \cdot A + C^T \cdot C = 0$$

- HSV: $\sigma_i = \sqrt{\lambda_i (P \cdot Q)}$

Theory

- Error bound: If the system is reduced to one with **k largest** HSV then

$$\|G - \hat{G}\|_{\infty} < 2(\sigma_{k+1} + \ldots + \sigma_n)$$



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Control Theory Methods

Transfer Function

- In Laplace Domain:

$$G_{\Sigma}(s) = C \cdot (sI - A)^{-1} \cdot B$$

Different methods

- Stable systems
  - In unstable systems something should be done with unstable poles.
- Hankel Norm Appr.
  - Produces an optimal solution
- Balanced Truncation Appr.
  - Most often used.
  - Faster than HNA.
  - Do not preserve the stationary state.
- Singular Perturbation Appr.
  - Preserve the stationary state.
- Frequency-weighted model reduction
  - $\|V(G - \hat{G})W\|_{\infty}$

SLICOT Library

- FORTRAN Code + Examples found at: `www.win.tue.nl/niconet`
  - European Community BRITE-EURAM III Thematic Networks Programme.
- Implements all methods:
  - Balanced Truncation Appr.
  - Singular Perturbation Appr.
  - Hankel Norm Appr.
  - Frequency-weighted MOR.
- Has a parallel version.
- Matlab has licensed SLICOT.

- Yet, the computational complexity is $O(N^3)$.
  - Limited to "small" systems:

Order	Time Serial	Time Parallel (4 processor)
600	60	25
1332	703	130
2450	4346	666
3906		2668